MATH2050C Assignment 3

Section 2.2 no. 10a, 11, 14b, 15d, 18a. Section 3.1 no. 5cd, 6a, 12, 14, 16, 17.

Deadline: Jan 28, 2025.

Hand in: Section 2.2 no. 10a, 15d. Section 3.1 no. 5c, 14. Suppl. Problems no. 6, 7.

Supplementary Problems

Problems 1-3 are for mathematical induction.

1. Prove that $5^{2n} - 1$ can be divided by 8 for all $n \in \mathbb{N}$.

2. Prove the generalized triangle inequality: For $a_1, a_2, \cdots, a_n \in \mathbb{R}$,
$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \ .$$

3. Prove the GM-AM Inequality: For $a_1, a_2, \cdots, a_n \geq 0$,
$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \ , \quad n \geq 1,$$
and equality in the inequality holds iff all a_j's are equal. Hint: Show it holds for $n = 2^k, k \geq 1$, first.

4. Show that for each positive number a and $n \geq 2$, there is a unique positive number b satisfying $b^n = a$. Suggestion: Use Binomial Theorem.

5. For $a > 0$ and $m, n \in \mathbb{N}$, define $a^{m/n} = (a^{1/n})^m$ and $a^{-m/n} = a^{-(m/n)}$. Show that (a) $a^{m/n} = (a^m)^{1/n}$ and (b) $a^{r+t} = a^r a^t, r, t \in \mathbb{Q}$.

6. Find the limit of $\{x_n\}, x_n = \frac{7n^2+3}{n^2-n-5}$.

7. Show that $\lim_{n\to\infty} n^{1/n} = 1$. Hint: Use Binomial Theorem.

See next page on limits of sequences.

Limits of Sequences

Mathematical induction is one of the most frequently used tools in analysis. It is apparently valid and, if you like, can be deduced from the more apparent fact, namely the Well-Ordering Property. Let us recall

The Well-Ordering Property. Every nonempty subset E of $\mathbb{N}$ has a least element in E.

This property is an axiom in nature. We do not have to prove it.

Principle of Mathematical induction. Let E be a subset of $\mathbb{N}$ satisfying (a) $1 \in \mathbb{N}$, and (b) $n+1 \in E$ whenever $n \in E$. Then $E = \mathbb{N}$.

Proof Assume on the contrary E is not $\mathbb{N}$. We apply the Well-Ordering Property to F, the complement of E, to find a least element $m \in F$, that is, $m \le n$ for all $n \in F$. Since m is the least element and $m > 1$ according to (a), $m-1$ is a natural number belonging to E. However, by (b) $m = (m-1)+1 \in F$, contradiction holds.

Now I give two examples of mathematical induction. These examples are important for later developments.

Bernoulli's Inequality. For $x > -1$,

$$(1+x)^n \ge 1 + nx \ .$$

Proof. Clearly the inequality holds when $n = 1$. Now assume it holds for n. Multiplying $(1+x)^n \ge 1+nx$ by $1+x > 0$, we have

$$(1+x)^{n+1} = (1+x)(1+x)^n \ge (1+x)(1+nx) = 1+(n+1)x+nx^2 \ge 1+(n+1)x \ ,$$

hence the inequality also holds at $n+1$. BY MI, it holds for all n.

Binomial theorem. For real a, b,

$$(a+b)^n = \sum_{k=0}^{n} C_k^n a^{n-k} b^k \ , \quad n \ge 1 \ .$$

Here $C_k^n = \frac{n!}{k!(n-k)!}$ and $0! = 1$.

Proof. The theorem is obvious when $n = 1$. Now assume n is true. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n} C_k^n a^{n-k}b^k \quad \text{by induction hypothesis} \\
&= \sum_{k=0}^{n} C_k^n a^{n-k+1}b^k + \sum_{k=0}^{n} C_k^n a^{n-k}b^{k+1} \\
&= \sum_{k=1}^{n} (C_k^n + C_{k-1}^n) a^{n-k+1}b^k + a^{n+1} + b^{n+1} \\
&= \sum_{k=1}^{n} C_k^{n+1} a^{n-k+1}b^k + a^{n+1} + b^{n+1} \\
&= \sum_{k=0}^{n+1} C_k^{n+1} a_k^{n+1-k}b^k \ .
\end{aligned}$$

So it holds also at $n+1$. By MI, the formula holds for all $n \in \mathbb{N}$.

Now we turn to sequences.

A sequence is a map from $\mathbb{N}$ to $\mathbb{R}$. However, we usually use notations like $\{x_n\}, \{a_k\}, \{b_m\}$ to denote sequences. For instance, the sequence given by $f(n) = 1/n$ is denoted by $\{1/n\}$. Given a sequence $\{x_n\}$, the number x is the limit of $\{x_n\}$ if for every $\varepsilon > 0$, there is some n_0 such that $|x_n - x| < \varepsilon$, for all $n \geq n_0$. Write it as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. A sequence is convergent if its limit exists. It is divergent if it does not have a limit. For instance, the sequences $\{n\}$ and $1 + (-1)^k\}$ are divergent.

Example 3.1 Show that $\lim_{n\to\infty} 1/n = x$. By the Archimedian property, for given $\varepsilon > 0$, there is some n_0 such that $0 < 1/n_0 < \varepsilon$. (Specifically we can choose n_0 to be any natural number greater than or equal to $[1/\varepsilon] + 1$ where $[a]$ means the integral part of a.) Now we have $|1/n - 0| = 1/n \leq 1/n_0 < \varepsilon$ for all $n \geq n_0$.

By the same argument taking $n_0 \geq [1/\varepsilon^{1/a}]+1$, one can show that for $a = p/q > 0$, $\lim_{n\to\infty} 1/n^a = 0$.

Proposition 3.1 Whenever there is some $\{c_n\}, c_n \to 0$ as $n \to \infty$ satisfying $|x_n - x| \leq c_n$, $\lim_{n\to\infty} x_n = x$.

Proof For $\varepsilon > 0$, there is some n_0 such that $|c_n| = c_n < \varepsilon$ for all $n \geq n_0$. But then we also have $|x_x - x| \leq c_n < \varepsilon$, done.

Example 3.2 Find the limit of $\{\sqrt{n+1} - \sqrt{n}\}$. We have

$$\sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n}) < 1/n^{1/2} \ .$$

Taking $c_n = 1/n^{1/2}$, by Proposition 3.1 $\lim_{n\to\infty}(n+1)^{1/2} - n^{1/2} = 0$.

Example 3.3 Show that for $b \in (0,1), \lim_{n\to\infty} b^n = 0$. Well, writing $b = 1/(1+c), c > 0$, by Bernoulli's inequality $(1+c)^n \geq 1+cn$. Therefore, $|b^n - 0| = b^n = 1/(1+c)^n \leq 1/(1+nc) < \frac{1/c}{n}$. By Proposition 3.1, $b^n \to 0$ as $n \to \infty$.

Example 3.4 Show that for $c > 0$, $\lim_{n\to\infty} c^{1/n} = 1$. First assume $c > 1$. Write $c^{1/n} = 1 + d_n, d_n > 0$. By Bernoulli's inequality, $(1 + d_n)^n \geq 1 + nd_n$. It follows that $c = (1 + d_n)^n \geq 1 + nd_n > nd_n$, that is, $0 < d_n < c/n$ so $d_n \to 0$ as $n \to \infty$. By Proposition 3.1, $|c^{1/n} - 1| = c^{1/n} - 1 = d_n < c/n$ implies $\lim_{n\to\infty} c^{1/n} = 1$. When $c \in (0, 1]$, simply observe that $1/c > 1$ and $\lim_{n\to\infty} c^{1/n} = 1/\lim_{n\to\infty} c^{-1/n} = 1$. (We will discuss this more in the next section on the limit theorems.)